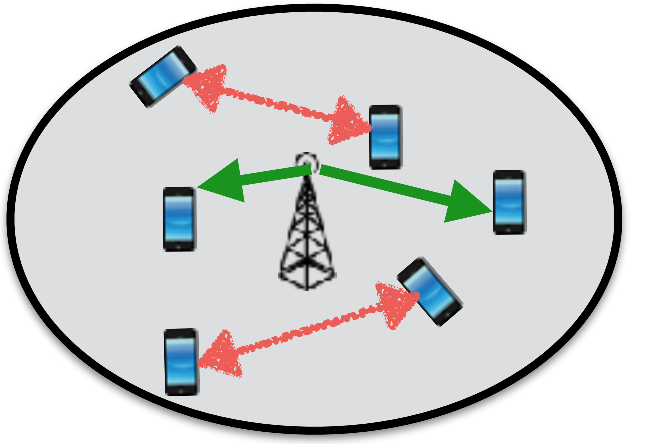
Interference Queuing Networks on Grids

Abishek Sankararaman, François Baccelli and Sergey Foss

Ad-Hoc Wireless Networks

Networks without a centralized infrastructure

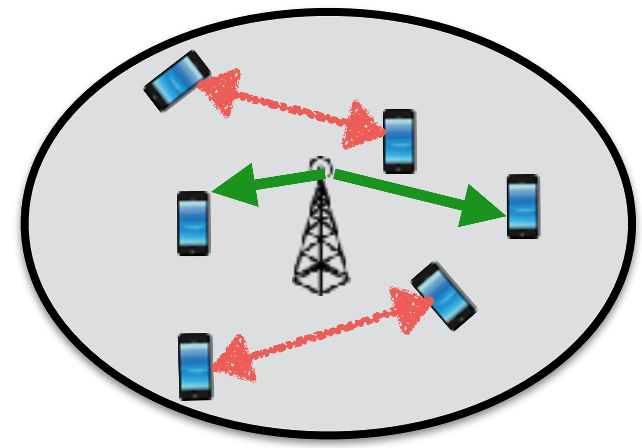


Examples -

1) Overlaid Device-to-Device (D2D) Networks

Ad-Hoc Wireless Networks

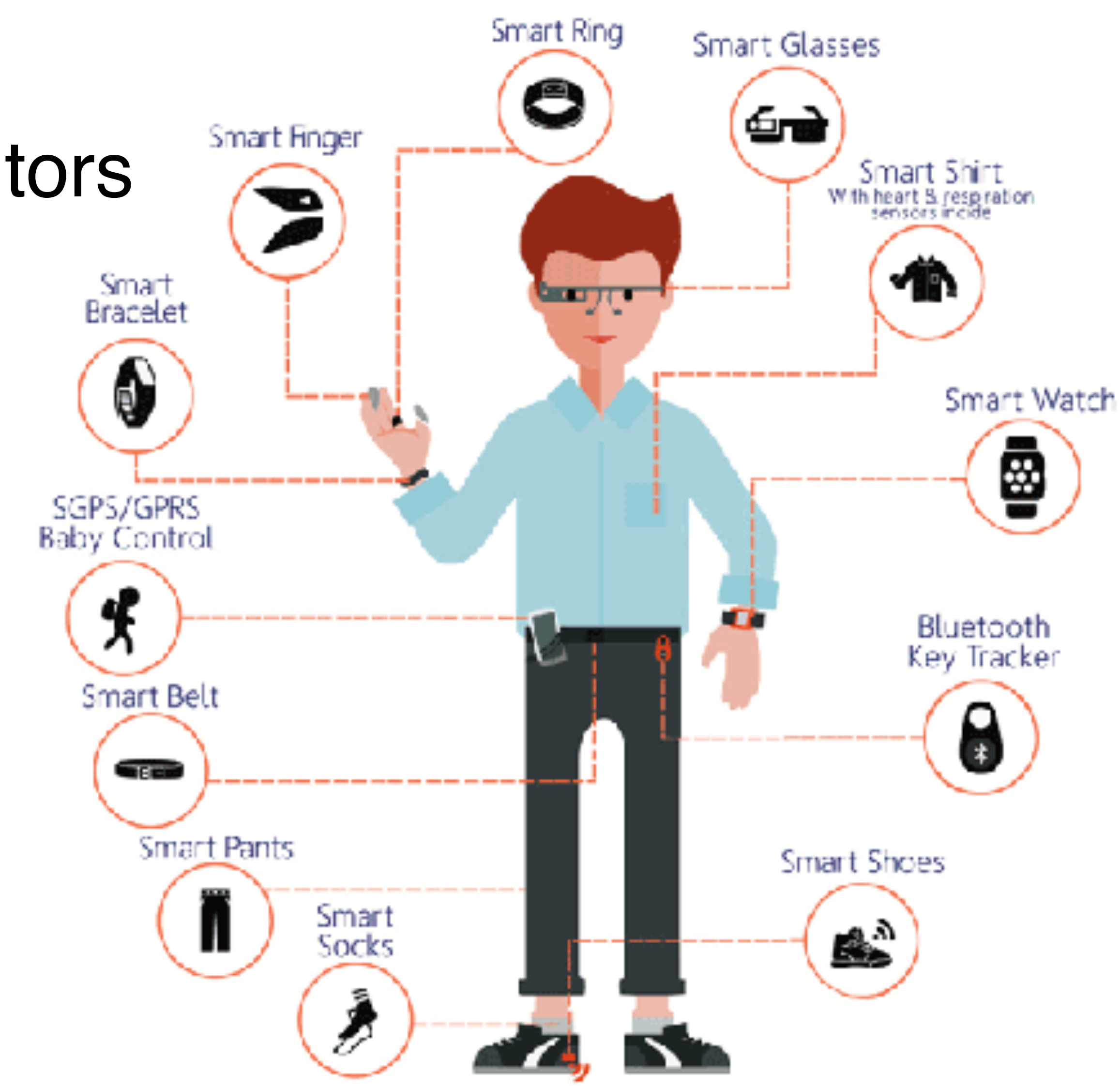
Networks without a centralized infrastructure



Examples -

1) Overlaid Device-to-Device (D2D) Networks

2) Internet of Things - Sensors and monitors

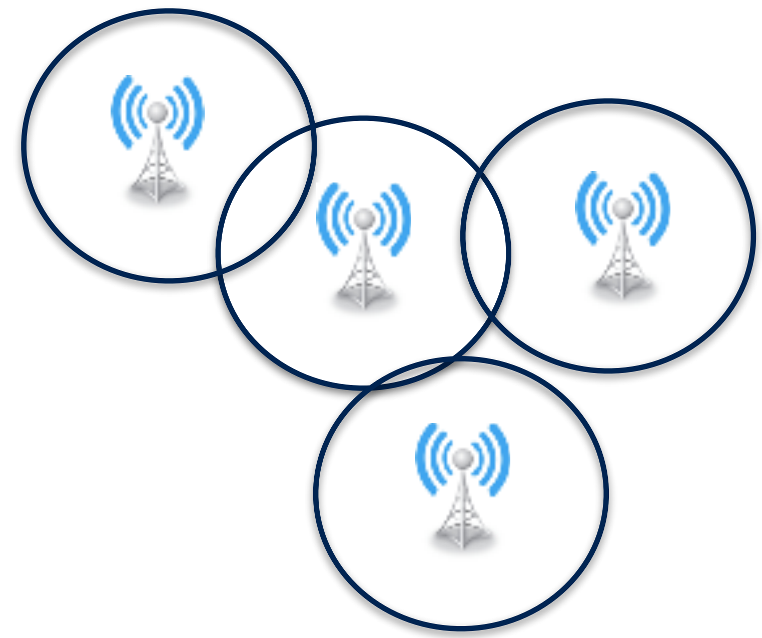


Wireless devices everywhere !

Spatio-Temporal Dynamics

Wireless Spectrum is a space-time shared resource

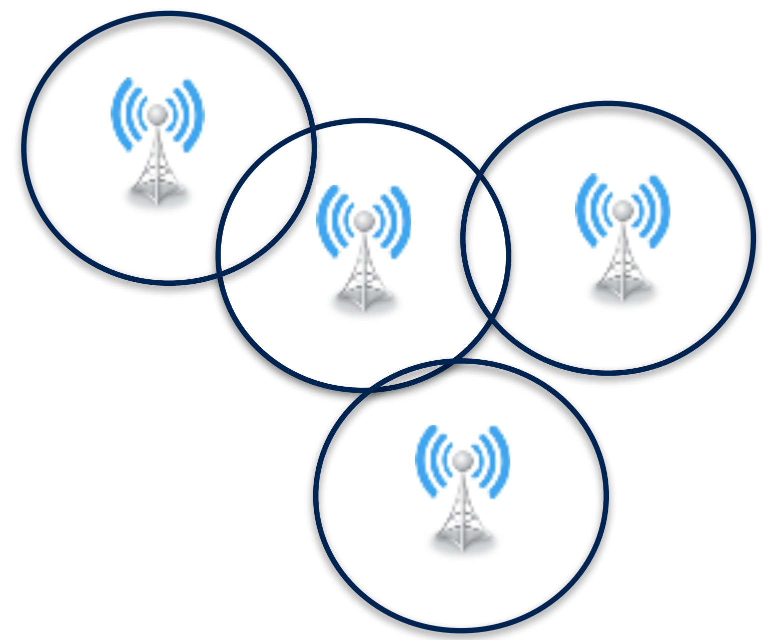
Spatial Component - Interference



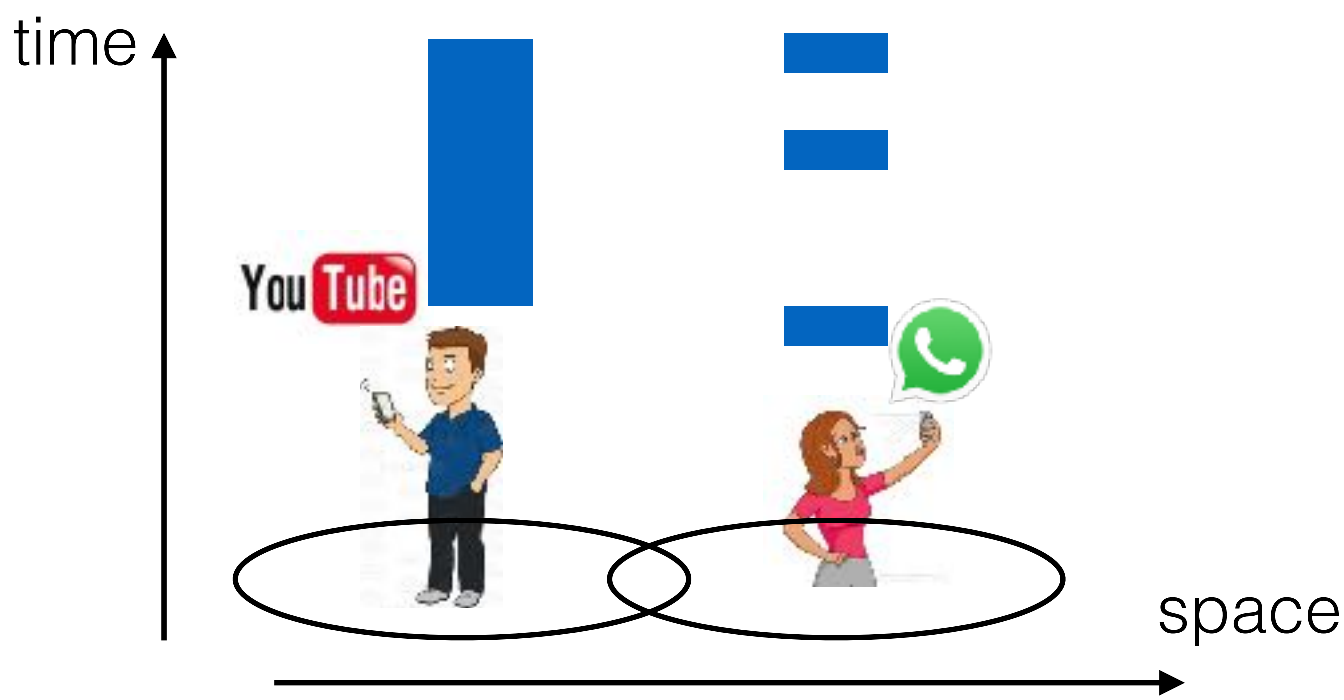
Spatio-Temporal Dynamics

Wireless Spectrum is a space-time shared resource

Spatial Component - Interference



Temporal Component - Traffic Patterns



Understanding the interplay of space-time interactions is crucial for design

Prior Work

Ad-hoc networks have been studied for a long time !

However, little is understood on the spatio-temporal interactions

1. Static spatial setting

[Gupta et al. 00][Baccelli et al. 03][Andrews et al. 07][De-Veciana et al. 08]
[Haenggi et al. 09]

(Does not precisely capture interactions through traffic arrivals)

2. Flow-based queuing models

[Bonald et al. 06][Srikant et al. 07][Shah et al, 09][Shakkottai et al. 07]
[De-Veciana et al. 08]

(Does not capture precisely, the information-theoretic interactions)

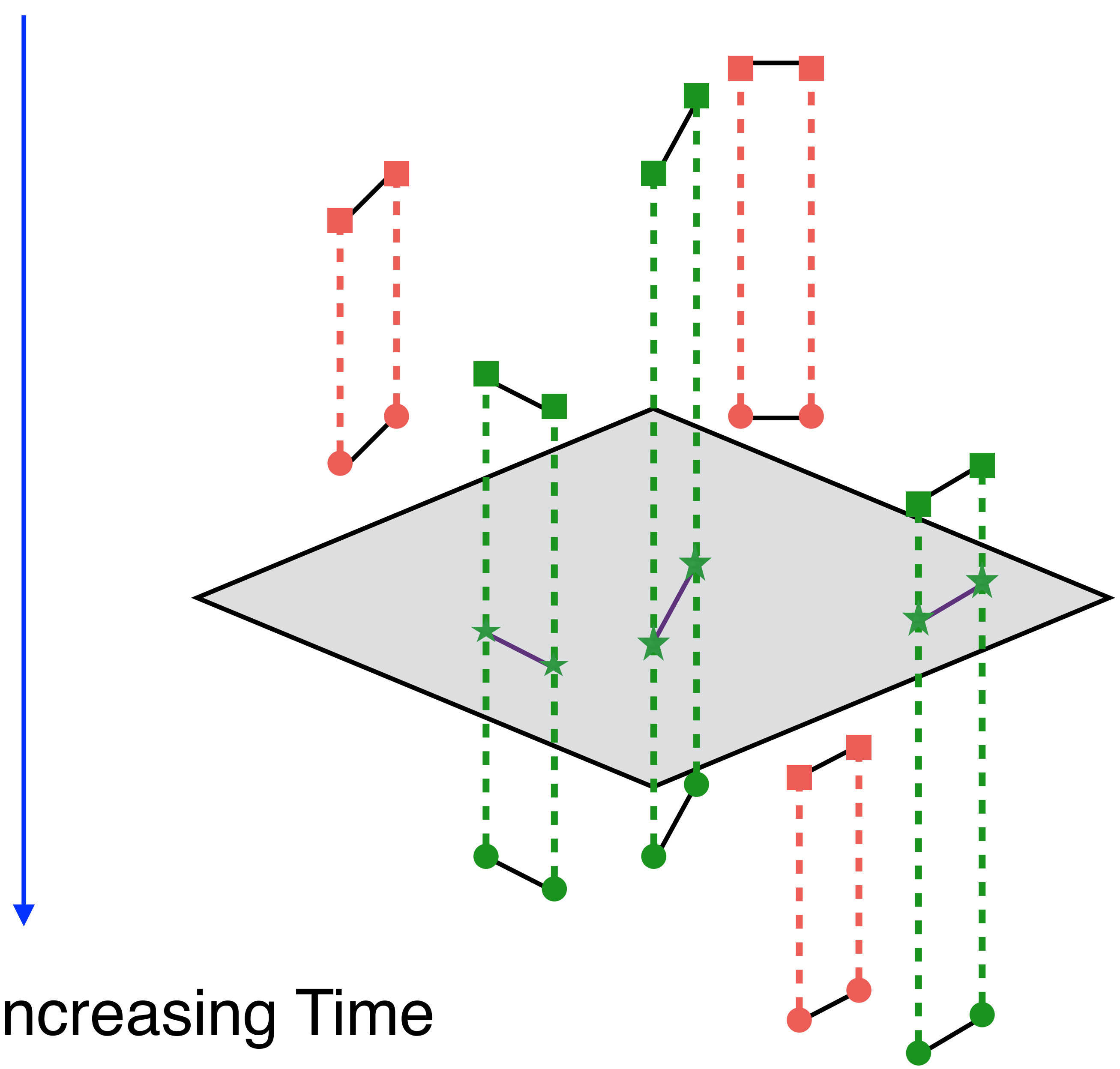
We provide a framework to capture interactions in space and time

Schematic - Spatial Birth-Death Process

Protocol - *A link transmits whenever they have a file by treating interference as noise*

Schematic - Spatial Birth-Death Process

Protocol - *A link transmits whenever they have a file by treating interference as noise*



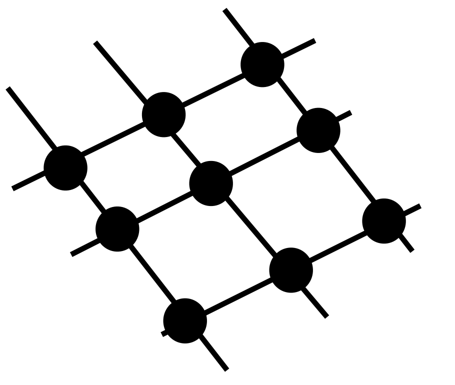
Increasing Time

When does this protocol “work” ?

Wireless Dynamics on Grids

Protocol - *A link transmits whenever they have a file by treating interference as noise*

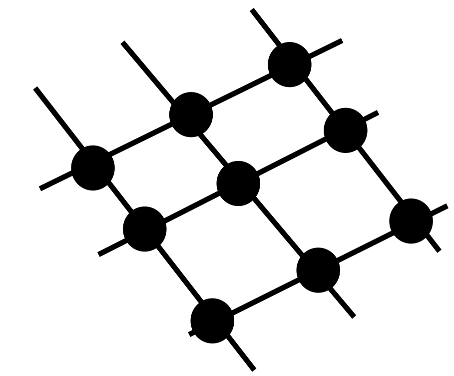
Discrete Space - d dimensional grid



Wireless Dynamics on Grids

Protocol - *A link transmits whenever they have a file by treating interference as noise*

Discrete Space - d dimensional grid



Each wireless link (Tx-Rx pair) is abstracted as a point

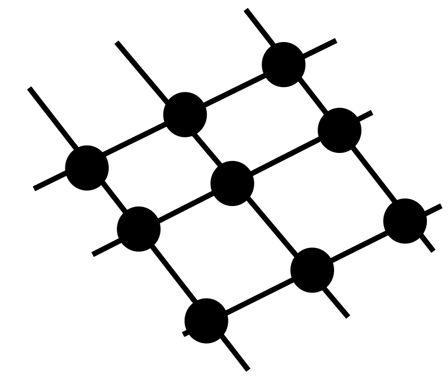
Links (points) 'arrive' uniformly in space and transmit

Links exit after completion of a file transfer

Wireless Dynamics on Grids

Protocol - *A link transmits whenever they have a file by treating interference as noise*

Discrete Space - d dimensional grid



Each wireless link (Tx-Rx pair) is abstracted as a point

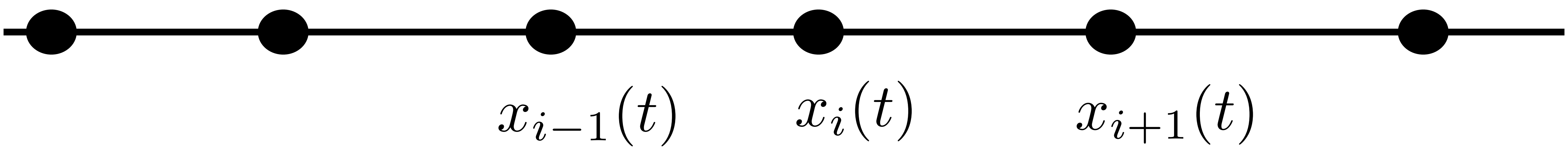
Links (points) ‘arrive’ uniformly in space and transmit

Links exit after completion of a file transfer

Instantaneous rate of transfer - Linearization of Shannon capacity formula

Interference as Noise

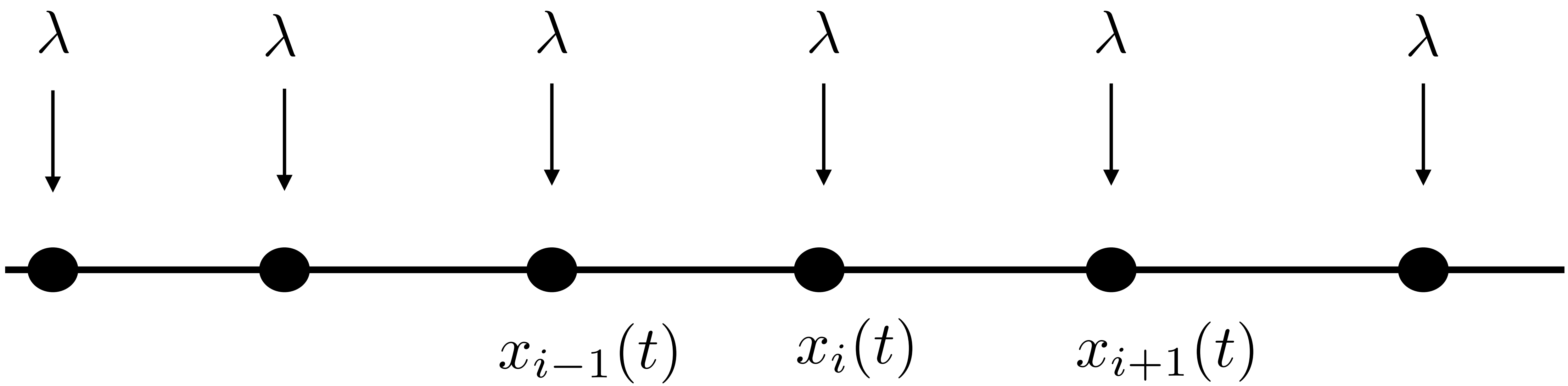
A warm up to the Model



$x_i(t) \in \mathbb{N}$ Number of links in cell $i \in \mathbb{Z}$ at time $t \geq 0$

$\{x_i(t)\}_{i \in \mathbb{Z}}$ Queue lengths at time $t \geq 0$

A warm up to the Model

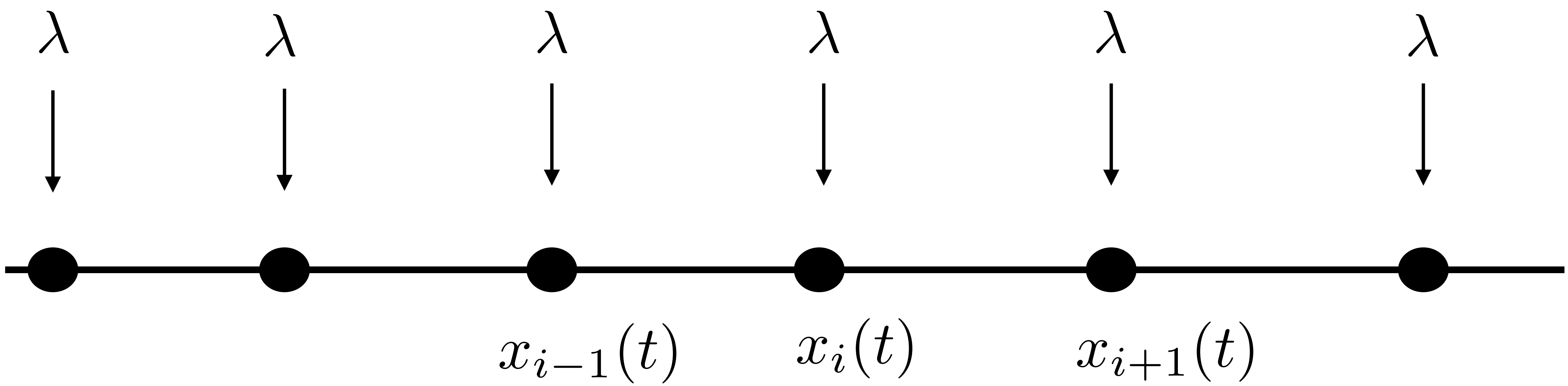


$x_i(t) \in \mathbb{N}$ Number of links in cell $i \in \mathbb{Z}$ at time $t \geq 0$

$\{x_i(t)\}_{i\in\mathbb{Z}}$ Queue lengths at time $t \geq 0$

Independent Poisson Arrivals

A warm up to the Model



$x_i(t) \in \mathbb{N}$ Number of links in cell $i \in \mathbb{Z}$ at time $t \geq 0$

$\{x_i(t)\}_{i \in \mathbb{Z}}$ Queue lengths at time $t \geq 0$

Independent Poisson Arrivals

Rate of departure from queue $i \in \mathbb{Z}$ at time t $\dfrac{x_i(t)}{x_{i-1}(t) + x_i(t) + x_{i+1}(t)}$

If ‘neighboring’ queues are large, instantaneous departure rate is small.

Rate of Departure - SIR

$\{x_i(t)\}_{i\in\mathbb{Z}^d} \in \mathbb{N}^{\mathbb{Z}^d}$ Queue Lengths

Rate of Departure - SIR

$\{x_i(t)\}_{i\in\mathbb{Z}^d} \in \mathbb{N}^{\mathbb{Z}^d}$ Queue Lengths

Interference Sequence $\{a_i\}_{i\in\mathbb{Z}^d}$

Rate of Departure - SIR

$\{x_i(t)\}_{i \in \mathbb{Z}^d} \in \mathbb{N}^{\mathbb{Z}^d}$ Queue Lengths

Interference Sequence $\{a_i\}_{i \in \mathbb{Z}^d}$

$a_i \geq 0 \ \forall i \in \mathbb{Z}^d$
$a_0 = 1$

Positivity

$a_i = a_{-i} \ \forall i \in \mathbb{Z}^d$

Symmetry

$L = \sup\{||i||_\infty : a_i > 0\} < \infty$

Finite Support

Interference at queue i - $\sum_{j \in \mathbb{Z}^d} a_j x_{i-j}(t)$

SIR at a customer in queue i at time t $\frac{1}{\sum_{j \in \mathbb{Z}^d} a_j x_{i-j}(t)}$

Rate of Departure - SIR

$\{x_i(t)\}_{i \in \mathbb{Z}^d} \in \mathbb{N}^{\mathbb{Z}^d}$ Queue Lengths

Interference Sequence $\{a_i\}_{i \in \mathbb{Z}^d}$

$a_i \geq 0 \; \forall i \in \mathbb{Z}^d$
$a_0 = 1$

Positivity

$a_i = a_{-i} \; \forall i \in \mathbb{Z}^d$

Symmetry

$L = \sup\{||i||_\infty : a_i > 0\} < \infty$

Finite Support

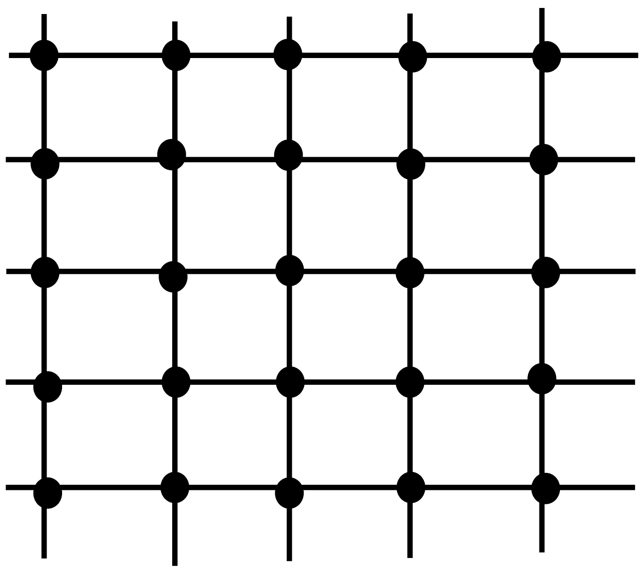
Interference at queue i - $\sum_{j \in \mathbb{Z}^d} a_j x_{i-j}(t)$

SIR at a customer in queue i at time t $\frac{1}{\sum_{j \in \mathbb{Z}^d} a_j x_{i-j}(t)}$

Rate of departure from any queue i at time t $\frac{x_i(t)}{\sum_{j \in \mathbb{Z}^d} a_j x_{i-j}(t)}$

Translation Invariant in Space

Interference Queueing Dynamics



$\{x_i(t)\}_{i\in\mathbb{Z}^d} \in \mathbb{N}^{\mathbb{Z}^d}$ Queue lengths at time $t \geq 0$

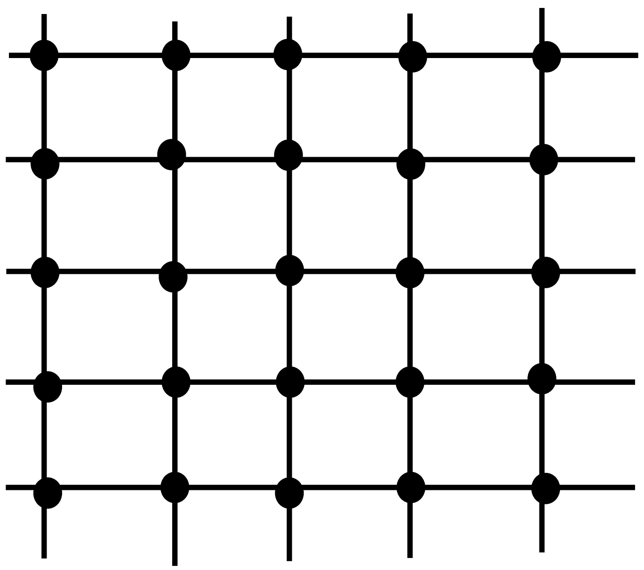
Independent rate λ Poisson arrivals

Rate of departure from queue $i \in \mathbb{Z}$ at time t $\quad \dfrac{x_i(t)}{\sum_{j\in\mathbb{Z}^d} a_j x_{i-j}(t)}$

If 'neighboring' queues are large, instantaneous departure rate is small.

In the toy example, $a_i = 1$ if $|i| \leq 1$ and $a_i = 0$ otherwise

Interference Queueing Dynamics



$\{x_i(t)\}_{i \in \mathbb{Z}^d} \in \mathbb{N}^{\mathbb{Z}^d}$ Queue lengths at time $t \geq 0$

Independent rate λ Poisson arrivals

Rate of departure from queue $i \in \mathbb{Z}$ at time t $\frac{x_i(t)}{\sum_{j \in \mathbb{Z}^d} a_j x_{i-j}(t)}$

If 'neighboring' queues are large, instantaneous departure rate is small.

Questions -

1) For what λ and $\{a_i\}_{i \in \mathbb{Z}^d}$, is the process $\{x_i(t)\}_{i \in \mathbb{Z}^d}$ 'stable' ?

2) Characterize the steady state ??

Connection with Related Models

Few Papers discuss Infinite Queuing Networks

1) Kelbert-Kontsevich-Rybko: *On Jackson Networks on Denumerable Graphs,* 1988.

2) Foss, Chernova: *On stability of polling models with infinite number of queues*, 1996.

3) Borovkov-Korshunov-Schassberger: *Ergodicity of a polling network with an infinite number of stations*, 1999.

4) Baccelli-Foss: *Poisson Hail on a Hot Ground*, 2011.

Similarities to *Interacting Particle System* (Liggett, 1985).
However, each particle (queue) has a countable number of states.

Main Results

1. Stability

If $\lambda \sum_{j \in \mathbb{Z}^d} a_j < 1$, then system is stable

Main Results

1. Stability

If $\lambda \sum_{j \in \mathbb{Z}^d} a_j < 1$, then system is stable

2. Moments

Let $\{y_i\}_{i \in \mathbb{Z}^d}$ be the minimal stationary solution to the dynamics

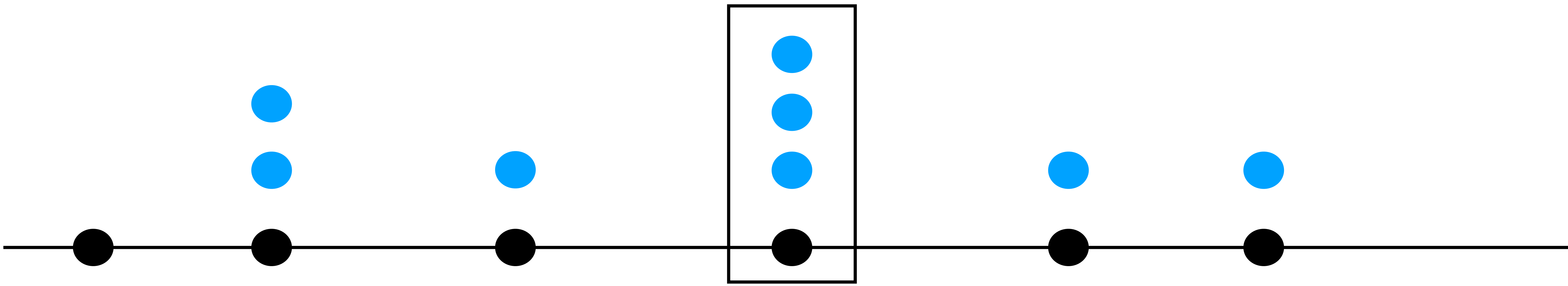
If $\lambda \sum_{j \in \mathbb{Z}^d} a_j < 1$, then $\mathbb{E}[y_0] = \dfrac{\lambda}{1 - \lambda \sum_{j \in \mathbb{Z}^d} a_j}$

If $\lambda \sum_{j \in \mathbb{Z}^d} a_j < \frac{2}{3}$ then $\mathbb{E}[y_0^2] < \infty$

[Shneer and Stolyar'18] established this for the entire stability range

In upcoming work with Sayan Banerjee, we show exponential moments exist in the entire range

Intuition



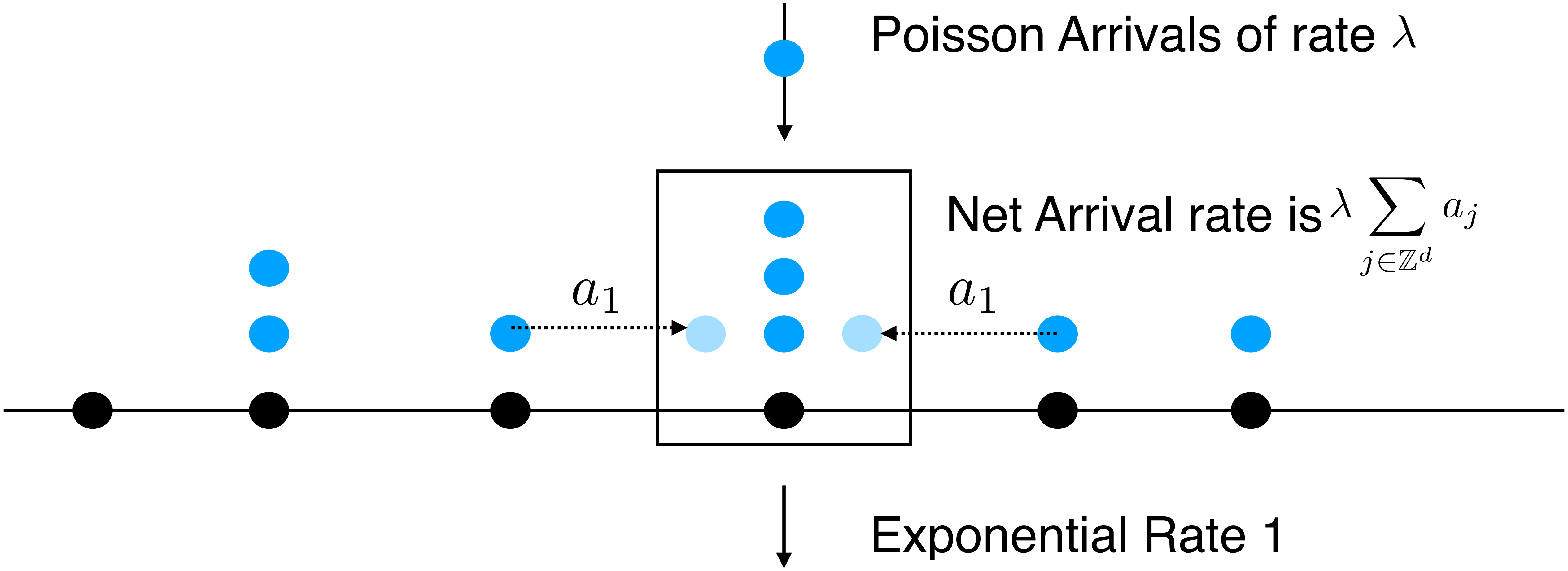
Consider any local maximum queue i, i.e. $x_i(t) = \max\{x_{i-j}(t) : a_j > 0\}$

Its instantaneous departure rate is $\dfrac{x_i(t)}{\sum_{j\in\mathbb{Z}^d} a_j x_{i-j}(t)} \geq \dfrac{1}{\sum_{j\in\mathbb{Z}^d} a_j}$

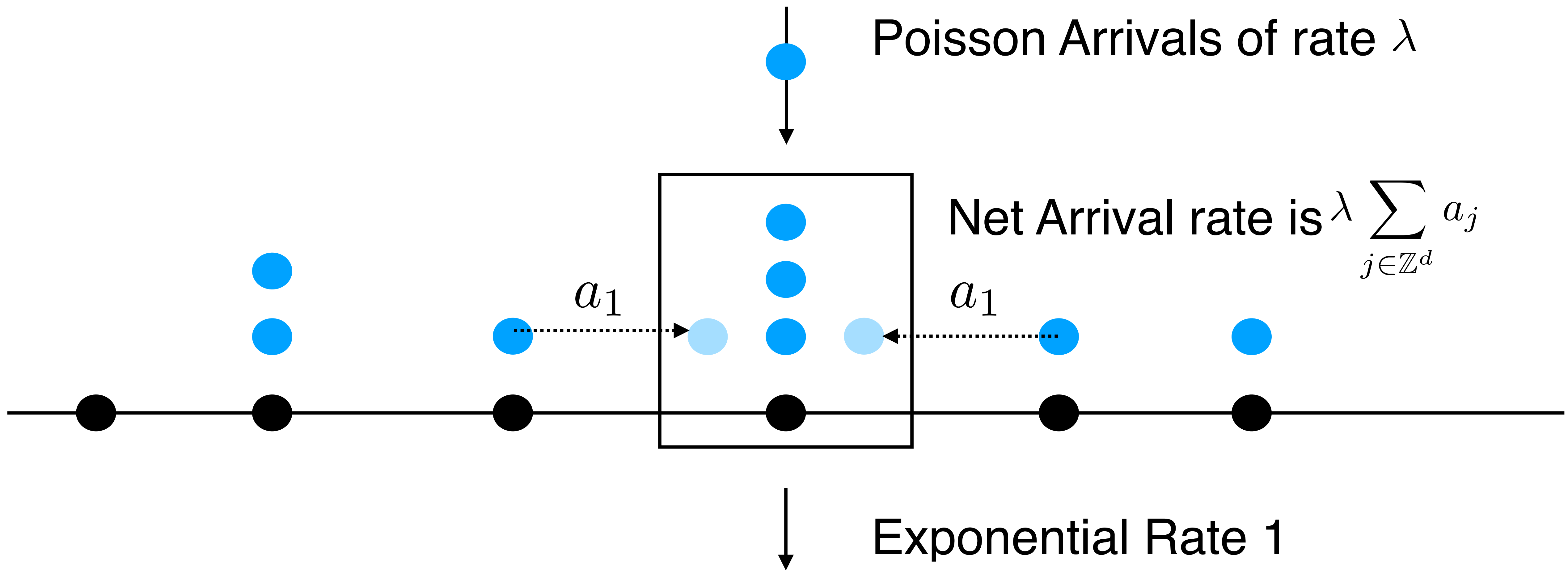
The arrival rate at every queue is λ

if $\lambda \sum_{j\in\mathbb{Z}^d} a_j < 1$, then this local maximum queue has negative drift

Intuition


Poisson Arrivals of rate λ
Net Arrival rate is $\lambda \sum_{j \in \mathbb{Z}^d} a_j$
a_1
a_1
Exponential Rate 1

Intuition



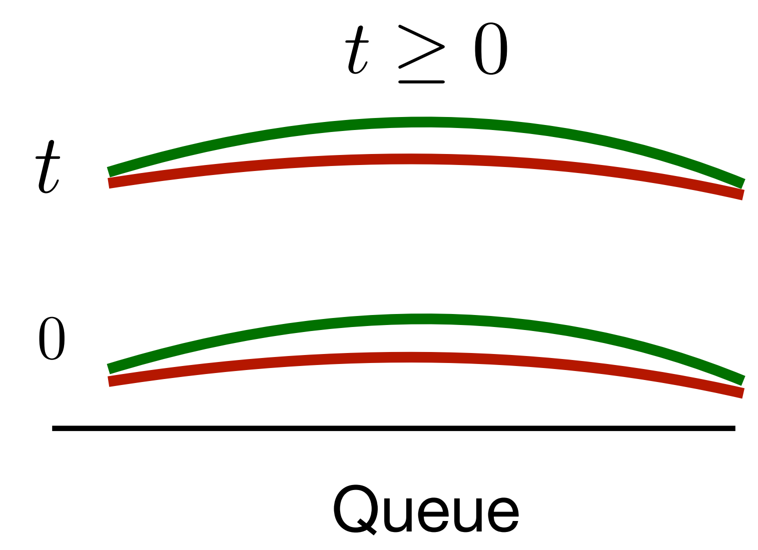
Stability - $$\lambda \sum_{j \in \mathbb{Z}^d} a_j < 1$$

Mean Queue Length - $$\frac{\lambda \sum_{j \in \mathbb{Z}^d} a_j}{1 - \lambda \sum_{j \in \mathbb{Z}^d} a_j} \cdot \frac{1}{\sum_{j \in \mathbb{Z}^d} a_j}$$

M/M/1 *Fraction of solid balls*

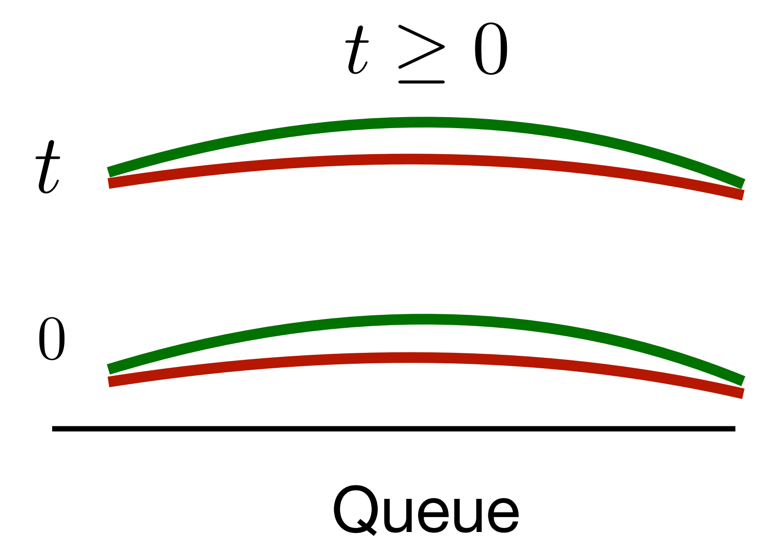
Monotonicity

If two initial conditions $\{x_i(0)\}_{i\in\mathbb{Z}^d}$ and $\{y_i(0)\}_{i\in\mathbb{Z}^d}$ s.t. for all $i \in \mathbb{Z}^d$ $x_i(0) \leq y_i(0)$, then there exists a coupling such that almost-surely $\forall t \geq 0$, $\forall i \in \mathbb{Z}^d$ $x_i(t) \leq y_i(t)$.



Monotonicity

If two initial conditions $\{x_i(0)\}_{i\in\mathbb{Z}^d}$ and $\{y_i(0)\}_{i\in\mathbb{Z}^d}$ s.t. for all $i \in \mathbb{Z}^d$ $x_i(0) \leq y_i(0)$, then there exists a coupling such that almost-surely $\forall t \geq 0$, $\forall i \in \mathbb{Z}^d$ $x_i(t) \leq y_i(t)$.



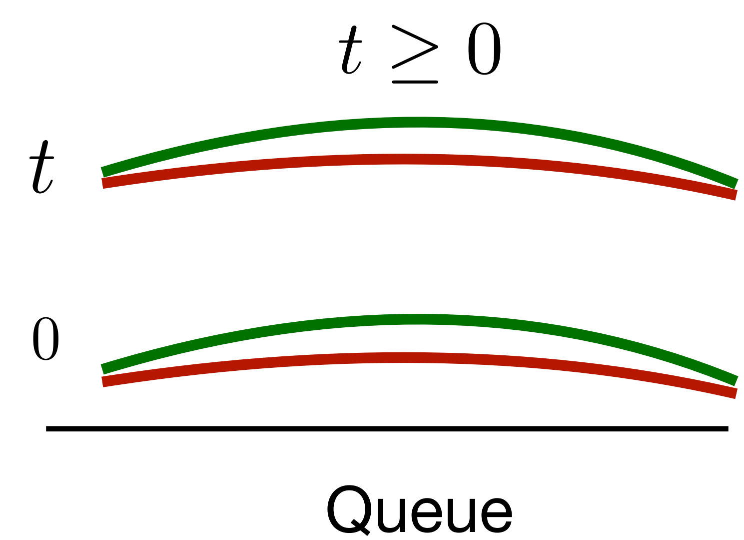
Proof Induction

Arrivals retain the ordering

Monotonicity

If two initial conditions $\{x_i(0)\}_{i\in\mathbb{Z}^d}$ and $\{y_i(0)\}_{i\in\mathbb{Z}^d}$ s.t. for all $i \in \mathbb{Z}^d$ $x_i(0) \leq y_i(0)$, then there exists a coupling such that almost-surely $\forall t \geq 0$, $\forall i \in \mathbb{Z}^d$ $x_i(t) \leq y_i(t)$.

<u>Proof</u> Induction

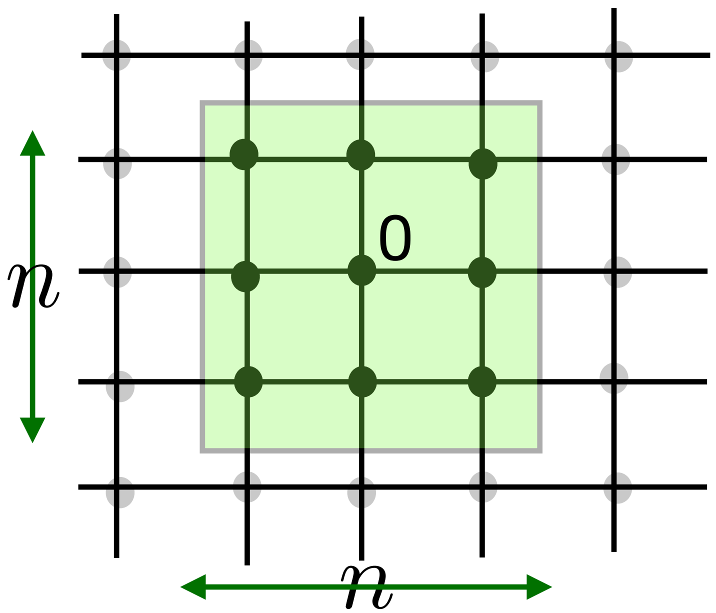


Arrivals retain the ordering

Two queues are equal - higher interference system has smaller departure

Unequal queues - Retains ordering as at-most one customer departs

Proof Steps



1. Consider a spatial truncation - finite dimensional

2. If $\lambda \sum_{j \in \mathbb{Z}^d} a_j < 1$ => Stability

Max queue length - Lyapunov function

3. Rate Conservation Principle

$\lambda \sum_{j \in \mathbb{Z}^d} a_j < 1$ => $\mathbb{E}[y_0^{(n)}] = \dfrac{\lambda}{1 - \lambda \sum_{j \in \mathbb{Z}^d} a_j} - o_n(1)$ *Tightness of* $\{y_0^{(n)}\}_{n \in \mathbb{N}}$

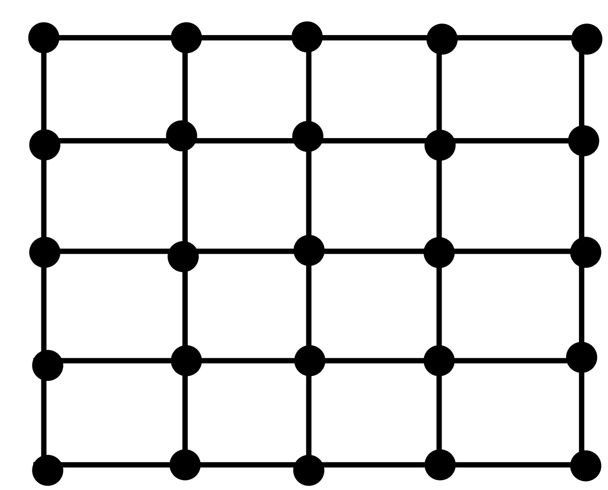
Main Proof Idea - Stability

Two systems on $B \subset \mathbb{Z}^d$ with the same dynamics.
All queues in $B^{\complement}$ are frozen without activity.

- $\{y_i(t)\}_{i \in B}$: the set B is a torus.

- $\{z_i(t)\}_{i \in B}$: the set B has boundary effects.

Interference is lower at the boundaries.



B

Main Proof Idea - Stability

Two systems on $B \subset \mathbb{Z}^d$ with the same dynamics.
All queues in $B^{\complement}$ are frozen without activity.

- $\{y_i(t)\}_{i \in B}$: the set B is a torus.

- $\{z_i(t)\}_{i \in B}$: the set B has boundary effects.

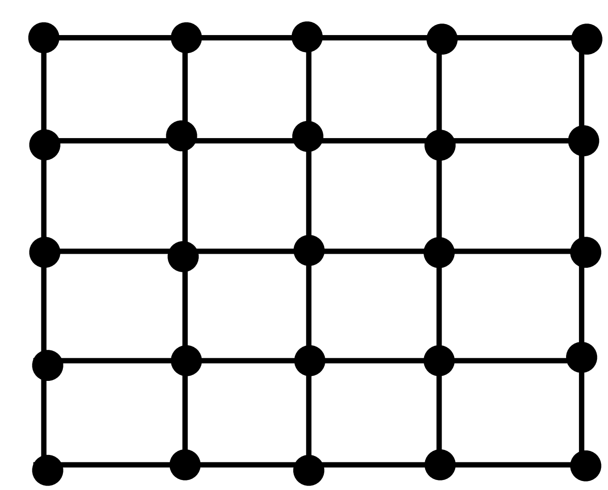
Interference is lower at the boundaries.

$\forall t \; \forall i \in B$

1) $x_i(t) \geq z_i(t)$

2) $y_i(t) \geq z_i(t)$

Monotonicity



B

Finite Torus System

$\{y_i(t)\}_{i\in B}$ process on a torus.

Theorem - If $\lambda \sum_{j\in\mathbb{Z}^d} a_j < 1$, then $\{y_i(t)\}_{i\in B}$ is Positive Recurrent and the stationary distribution possess exponential moments. Furthermore, the mean queue length satisfies $\mathbb{E}[y_0(t)] = \dfrac{\lambda}{1-\lambda\sum_{j\in\mathbb{Z}^d} a_j}$

Finite Torus System

$\{y_i(t)\}_{i\in B}$ process on a torus.

<u>Theorem</u> - If $\lambda \sum_{j\in\mathbb{Z}^d} a_j < 1$, then $\{y_i(t)\}_{i\in B}$ is Positive Recurrent and the stationary distribution possess exponential moments. Furthermore, the mean queue length satisfies $\mathbb{E}[y_0(t)] = \dfrac{\lambda}{1-\lambda\sum_{j\in\mathbb{Z}^d} a_j}$

<u>Proof Idea of Stability</u>

$$\frac{d}{dt} y_i = \lambda - \frac{y_i}{\sum_{j\in\mathbb{Z}^d} a_j y_{(i-j)/B}(t)}$$

Fluid scale equation

Consider the maximal queue $i^*(t) := \arg\max_{i\in B} y_i(t)$

$$\frac{d}{dt} y_{i^*(t)} = \lambda - \frac{y_{i^*(t)}}{\sum_{j\in\mathbb{Z}^d} a_j y_{i^*(t)-j}(t)}$$

This has negative drift

$$\leq \lambda - \frac{1}{\sum_{j\in\mathbb{Z}^d} a_j} \quad < -\epsilon$$

Can upper bound by a stable Single server queue.

Finite Torus System

Rate Conservation - "On Average what comes in is what goes out".

For Ex. $$\lambda = \mathbb{E}\left[\frac{y_0(t)}{\sum_{j\in\mathbb{Z}^d} a_j y_{j/B}(t)}\mathbf{1}_{y_0(t)>0}\right]$$

Avg arrival rate equals avg departure rate.

<u>Key Idea:</u>

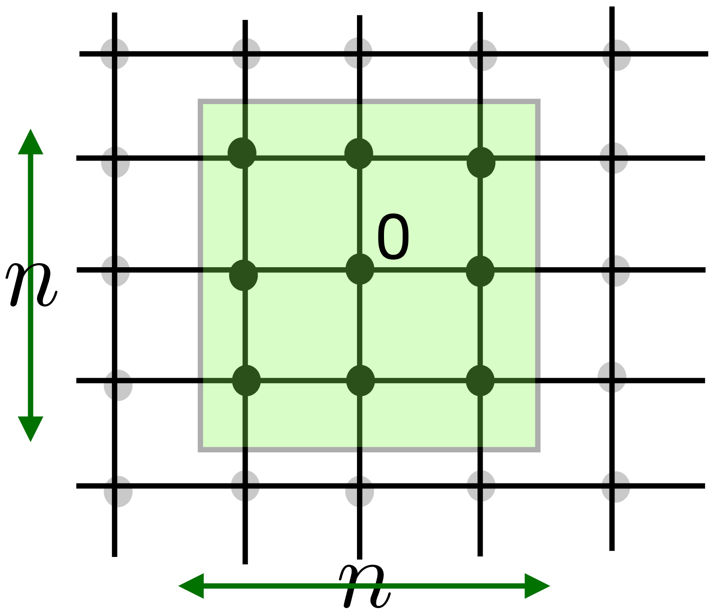
Consider $I(t) := y_0(t)\sum_{j\in\mathbb{Z}^d} a_j y_j(t)$ in stationarity and solve $\frac{d}{dt}\mathbb{E}[I(t)] = 0$

Average increase due to arrivals - $\lambda + \lambda(\sum_{j\in\mathbb{Z}^d} a_j)\mathbb{E}[y_0(t)]$

Average decrease due to departures - $\mathbb{E}[y_0(t)]$

Equating the two yields $\mathbb{E}[y_0(t)] \in \left\{\frac{\lambda}{1-\lambda\sum_{j\in\mathbb{Z}^d} a_j}, \infty\right\}$

Proof Steps



1. Consider a spatial truncation - finite dimensional

2. If $\lambda \sum_{j \in \mathbb{Z}^d} a_j < 1$ => Stability

   Max queue length - Lyapunov function

3. Rate Conservation Principle

$$\lambda \sum_{j \in \mathbb{Z}^d} a_j < 1 \quad \Rightarrow \quad \mathbb{E}[y_0^{(n)}] = \frac{\lambda}{1 - \lambda \sum_{j \in \mathbb{Z}^d} a_j} - o_n(1)$$

Tightness of $\{y_0^{(n)}\}_{n \in \mathbb{N}}$

4. Switch of limits in time and space *Coupling from the past*

5. Monotone Convergence to yield the moment formula

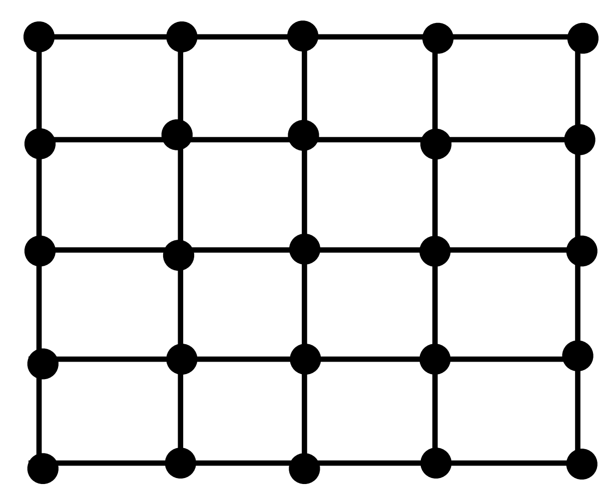
Coupling From the Past

$\{z_i(t)\}_{i\in B}$, process where the set B has boundary effects.

Monotonicity => $x_i(t) \geq z_i(t)$ and $y_i(t) \geq z_i(t)$

Thus $\mathbb{E}[z_0(t)] \leq \dfrac{\lambda}{1-\lambda\sum_{j\in\mathbb{Z}^d} a_j}$ ***Uniformly in the size of*** B

Consider $B_n \nearrow \mathbb{Z}^d$ and corresponding stationary $z_0^{(n)}(0)$

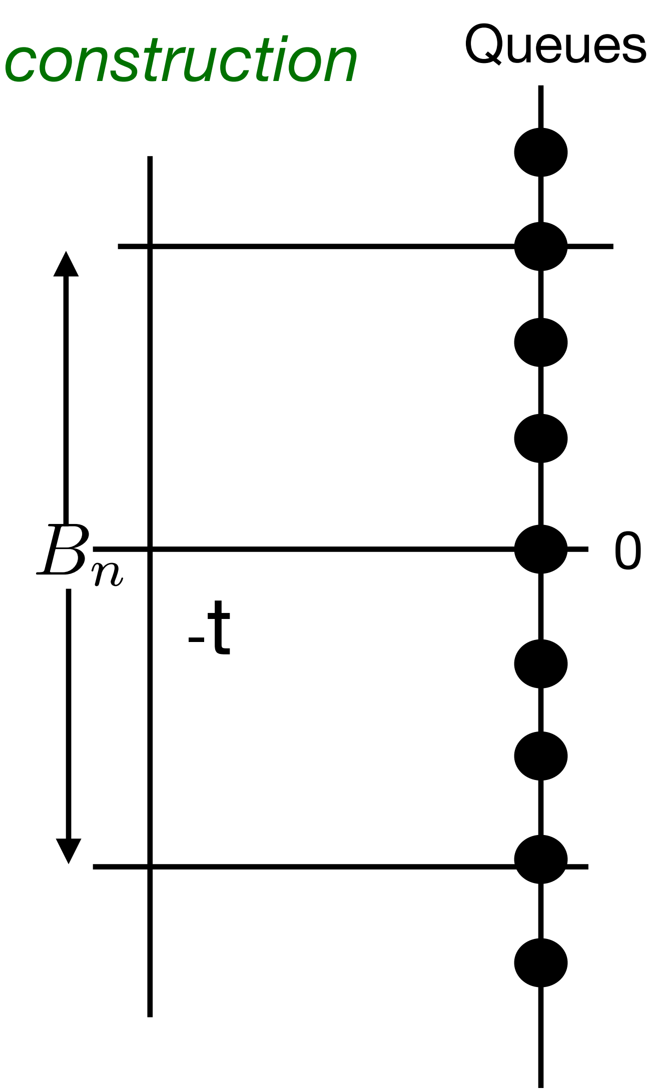


B

Coupling From the Past

Let $B_n \nearrow \mathbb{Z}^d$. $z^{(n)}_{0,t}(0)$ - the queue length of queue 0 at time 0, when the truncated B_n system is started empty at time -t.

Notice $\forall t \geq 0 \quad \lim_{n\to\infty} z^{(n)}_{0,t}(0) = x_{0,t}(0)$ *Corollary of the construction*



Monotonicity =>

$\lim_{t\to\infty} z^{(n)}_{0,t} := z^{(n)}_{0,\infty}$ and $\lim_{n\to\infty} z^{(n)}_{0,\infty} := z^{(\infty)}_{0,\infty}$ a.s.

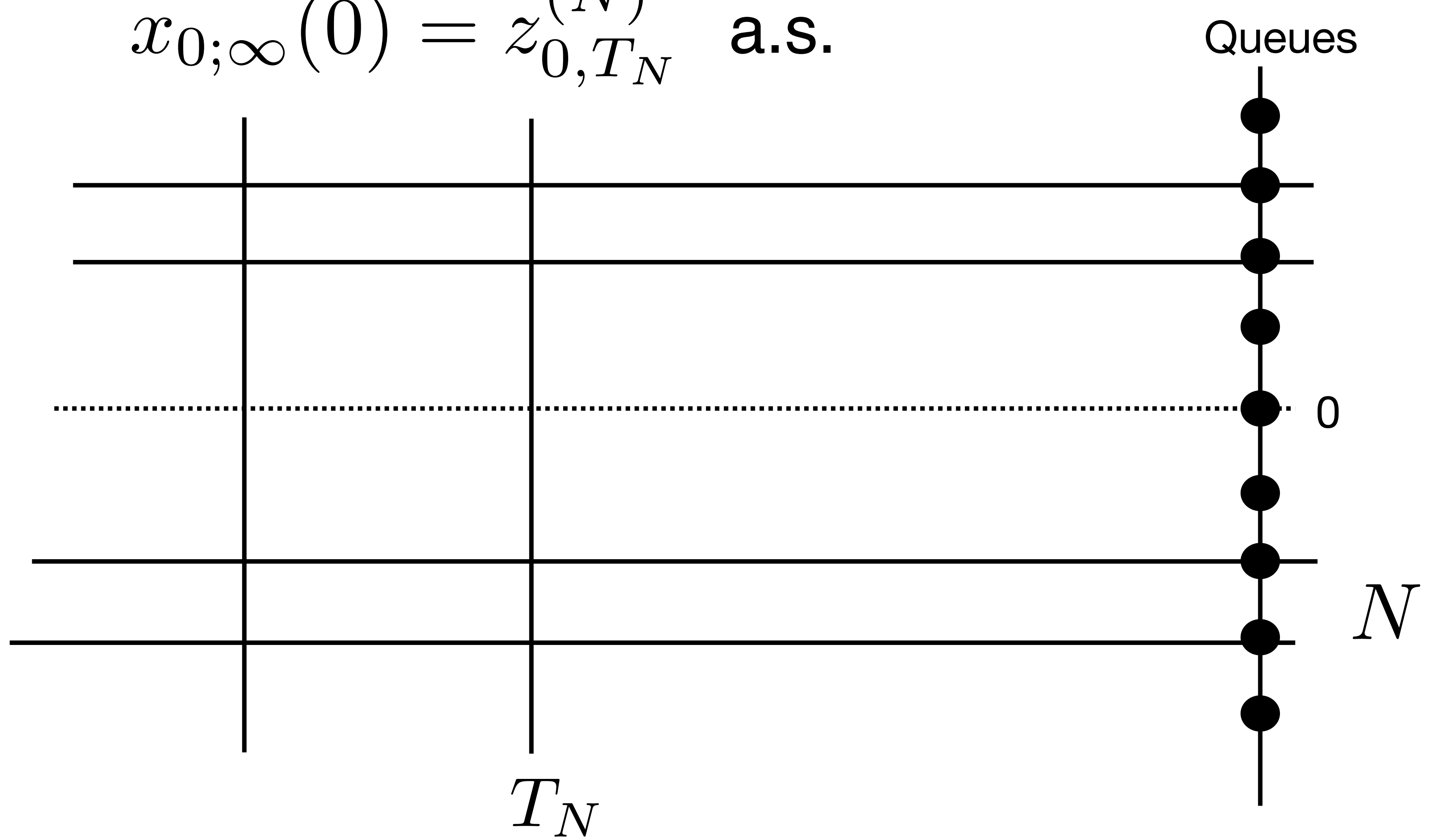
We know $\sup_{n\in\mathbb{N}} \mathbb{E}[z^{(n)}_{0,\infty}] \leq \dfrac{\lambda}{1-\lambda\sum_{j\in\mathbb{Z}^d} a_j}$

thus, $\mathbb{E}[z^{(\infty)}_{0,\infty}] < \infty$

Coupling From the Past

<u>Lemma</u> - If $\lambda \sum_{j \in \mathbb{Z}^d} a_j < 1$, then $\exists N \in \mathbb{N}$ and $\exists T_N < \infty$ random such that

$$x_{0;\infty}(0) = z^{(N)}_{0,T_N} \quad \text{a.s.}$$



We know $\sup_{n \in \mathbb{N}} \mathbb{E}[z^{(n)}_{0,\infty}] \leq \dfrac{\lambda}{1 - \lambda \sum_{j \in \mathbb{Z}^d} a_j}$. Thus $\mathbb{E}[x_{0,\infty}(0)] \leq \dfrac{\lambda}{1 - \lambda \sum_{j \in \mathbb{Z}^d} a_j}$

$\blacksquare$

Large Initial Conditions

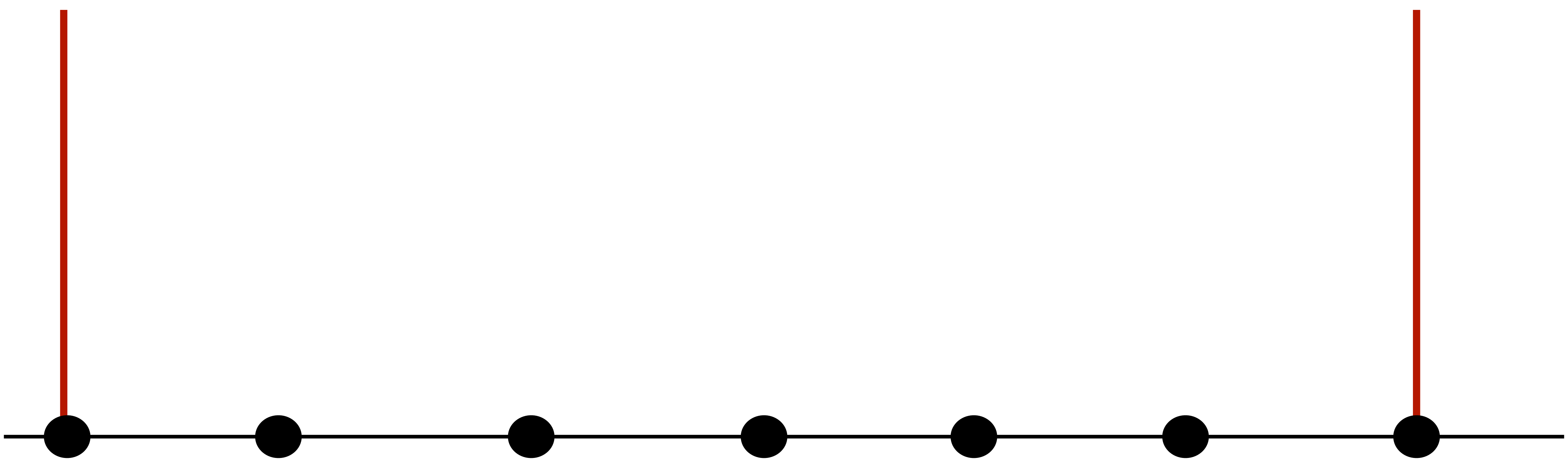
Theorem

For every λ , there exists a probability distribution on $\mathbb{N}$ such that if the initial condition is $\{x_i(0)\}_{i\in\mathbb{Z}^d}$ i.i.d. from this distribution, then $\forall i \in \mathbb{Z}^d$, $\lim_{t\to\infty} x_i(t) = \infty$ almost-surely.

Large Initial Conditions

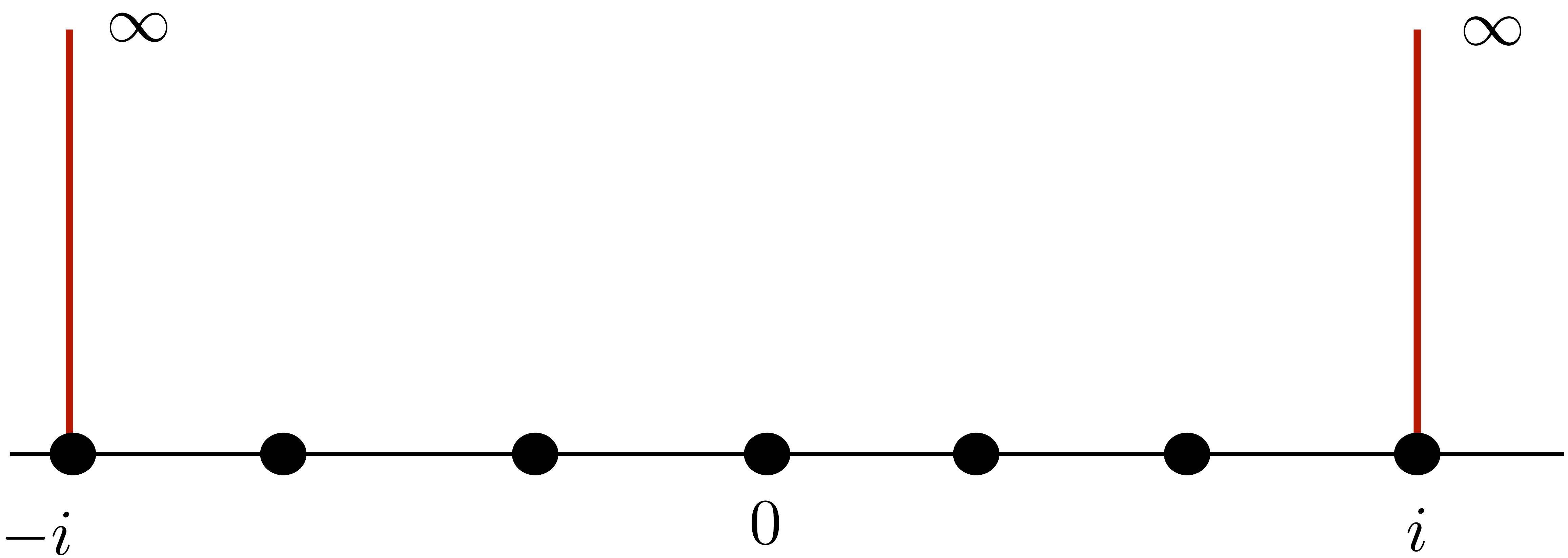
Theorem

For every λ , there exists a probability distribution on $\mathbb{N}$ such that if the initial condition is $\{x_i(0)\}_{i\in\mathbb{Z}^d}$ i.i.d. from this distribution, then $\forall i \in \mathbb{Z}^d$, $\lim_{t\to\infty} x_i(t) = \infty$ almost-surely



If “large” frozen boundary is present, then stationary queue length at 0 is also “large” with “high probability”

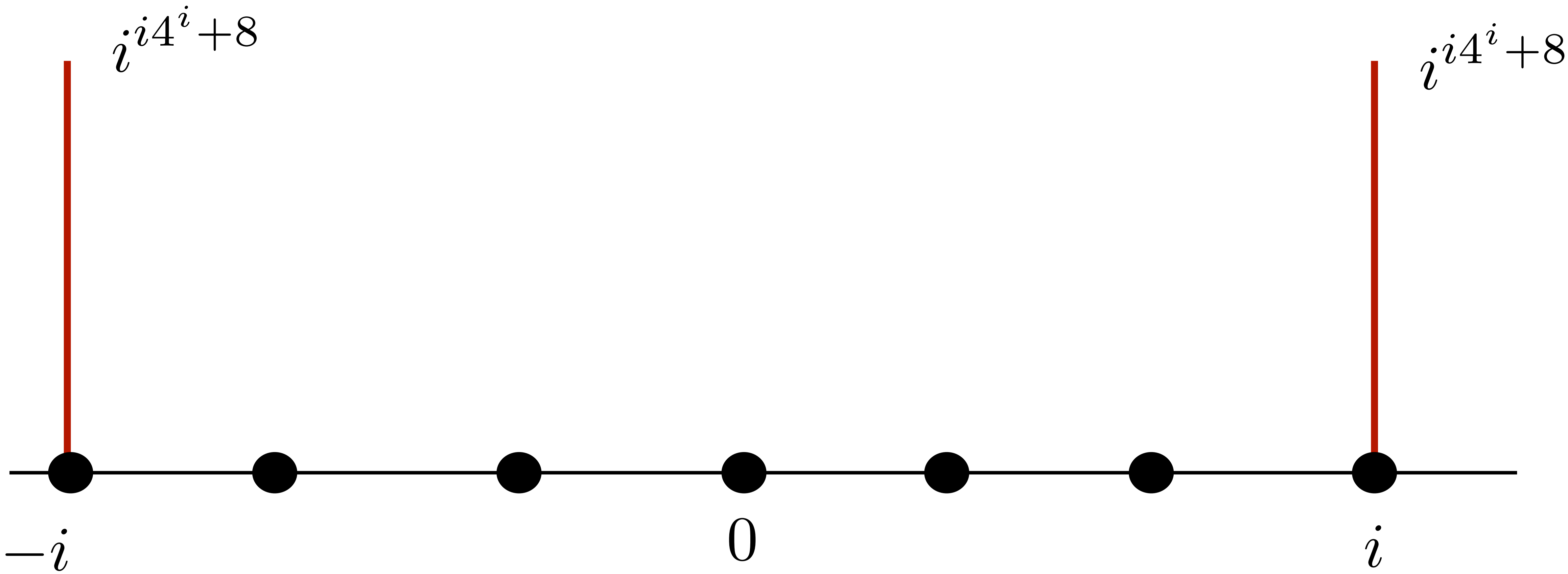
Convergence to Stationary Solutions



$$\exists (t_i)_{i\in\mathbb{N}} \text{ s.t. } t_i \to \infty \text{ s.t. } \mathbb{P}[x_0(t_i) < i] \leq i^{-4}$$

Because of the infinite barrier, all queues diverge to infinity at a linear rate

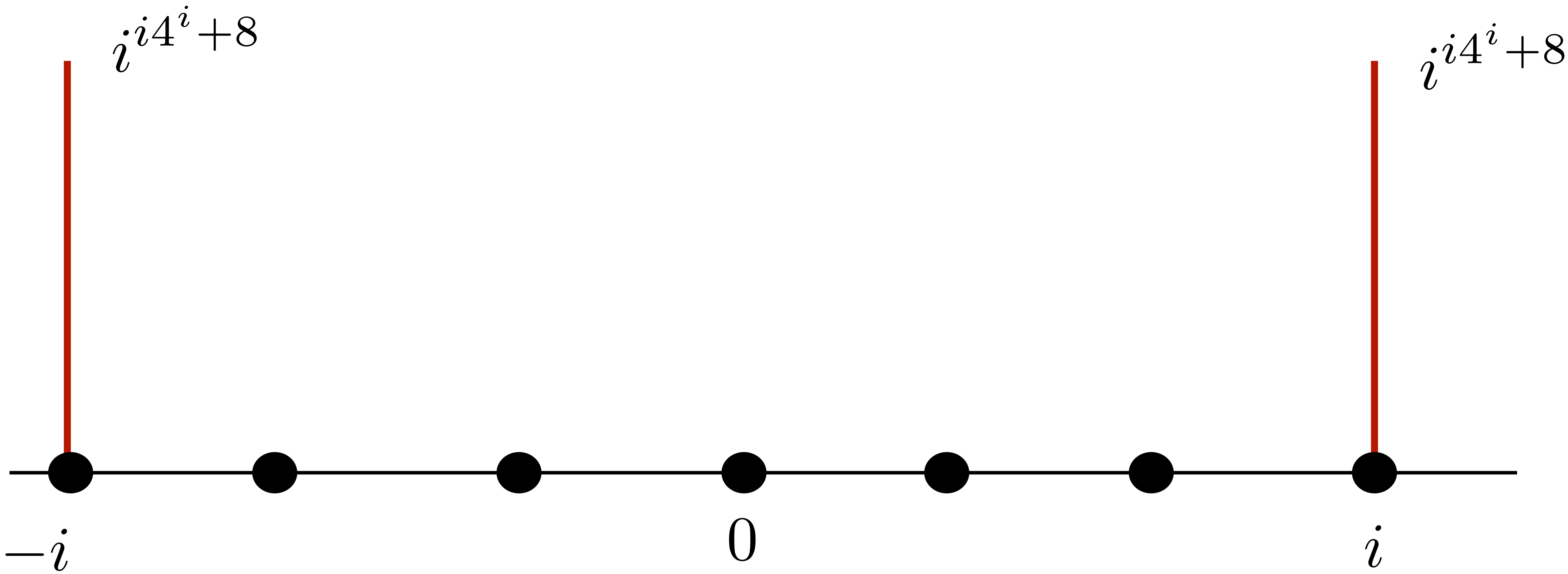
Convergence to Stationary Solutions



$$\exists (t_i)_{i \in \mathbb{N}} \text{ s.t. } t_i \to \infty \text{ s.t. } \mathbb{P}[x_0(t_i) < i] \leq i^{-3}$$

Since interested only in finite time t_i, can bring down the barrier to a finite value at a small penalty in probability

Convergence to Stationary Solutions



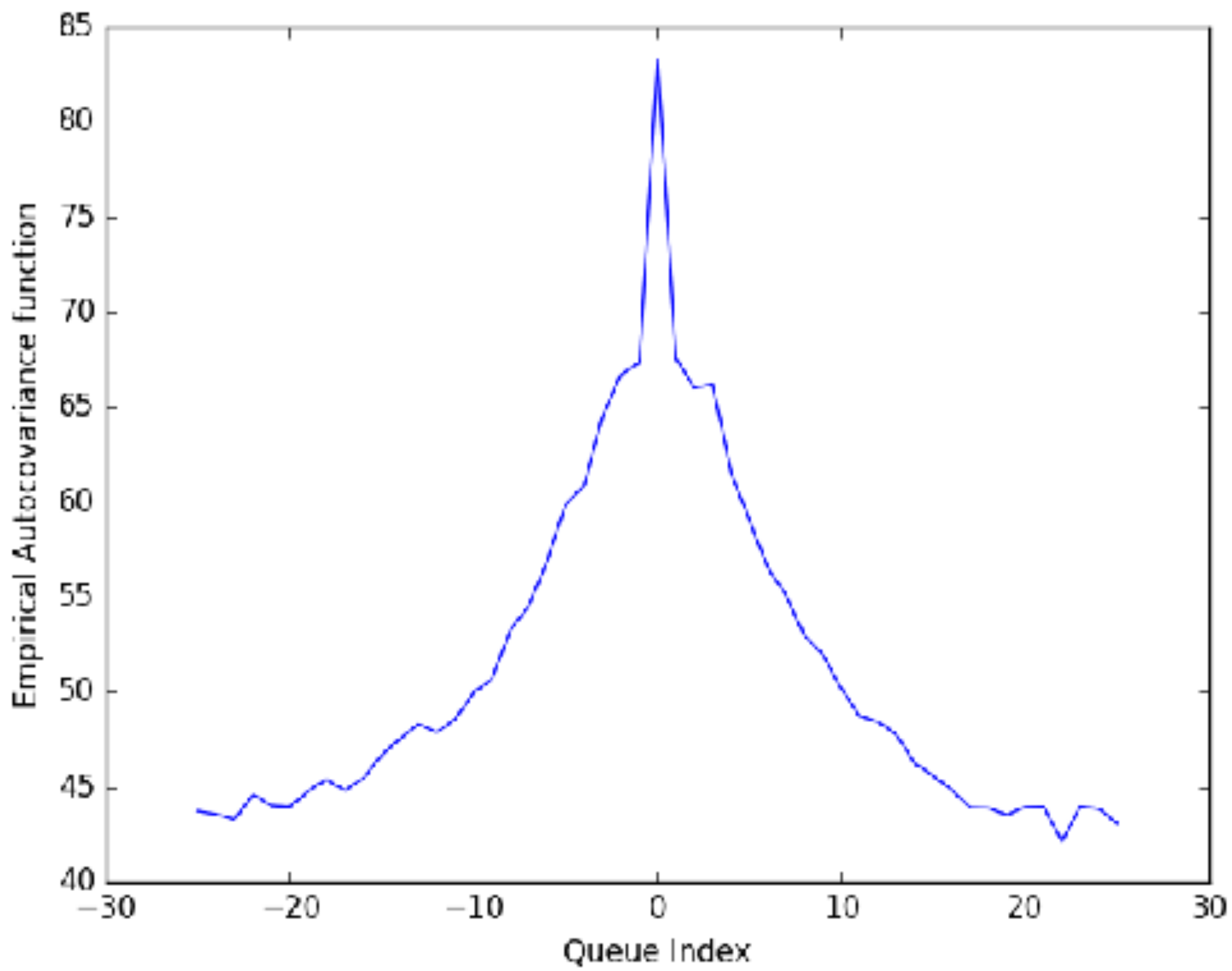
$$\exists (t_i)_{i\in\mathbb{N}} \text{ s.t. } t_i \to \infty \text{ s.t. } \mathbb{P}[x_0(t_i) < i] \leq {\color{red} i^{-3}}$$

Since interested only in finite time t_i, can bring down the barrier to a finite value at a small penalty in probability

Borel-Cantelli to conclude the proof

Open Questions

How do correlations $k \to \mathbb{E}[y_0 y_k] - (\mathbb{E}[y_0]^2)$ decay ?



d=1, n=51

$\lambda = 0.1419, \lambda_c = 1/7$

No propagation of chaos even in an infinite system !

Open Questions

Uniqueness of Stationary Solution

Existence/construction of other non-degenerate stationary solutions ?

Convergence to Stationary Solution

Do other initial conditions apart from all empty converge to a stationary limit ?

Prediction of bad outage events propagating from ‘far out’ in space

Thank You

Related Papers -

1) *Interference Queueing Networks on Grids* with F. Baccelli and S. Foss,
In Annals of Applied Probability, to appear
https://arxiv.org/abs/1710.09797

2) *Spatial Birth-Death Wireless Networks,* with F. Baccelli
In IEEE Transactions on Information Theory, 2017.
https://arxiv.org/abs/1604.07884